Rowan University
Department of Electrical and Computer Engineering

Estimation and Detection Theory
Fall 2013

Practice EXAM Solution

- This is a closed book exam.
- One letter-size sheet is allowed.
- There are 5 problems in the exam.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and show all relevant work. Your grade on each problem will be based on our assessment of your level of understanding as reflected by what you have written in the space provided.
- Please be neat and box your final answer, we cannot grade what we cannot decipher.

Name

Problem 1

What is

(1) an unbiased estimator?

(2) an efficient estimator?

(3) a MVU estimator?

(4) a BLUE estimator?

(5) a white noise

Solution

(1) An estimator is unbiased if *on the average* the estimator will yield the true value of the unknown parameter. Mathematically,

$$\hat{\theta} \text{ is unbiased if } E[\hat{\theta}] = \theta. \tag{1}$$

(2) An estimator is efficient if it is unbiased and attains the CRLB. It is called efficient because it efficiently uses the data.

(3) an MVU estimator is an estimator that is unbiased and has minimum variance among all unbiased estimators.

(4) a BLUE estimator is the Best Linear Unbiased Estimator. The BLUE has the form

$$\hat{\theta} = \sum_{i=0}^{N-1} a_n x[n], \tag{2}$$

where the a_n's are determined so that the estimator $\hat{\theta}$ is unbiased and has minimum variance.

(5) A white noise is a sequence, $\{w[n]\}$, such that all its samples are uncorrelated, i.e., $E[w[n][m]] = 0$, for $m \neq n$.

Problem 2

The data $x[n] = Ar^n + w[n]$ for $n = 0, 1, \cdots, N-1$ are observed, where $w[n]$ is WGN with variance σ^2 and $r > 0$ known .

(1) Find the CRLB for A

(2) Show that an efficient estimator exists and find its variance (using the CRLB theorem).

(3) Write the linear model for this data in the form $\mathbf{x} = HA + \mathbf{w}$.

(4) Using (3), find the MVU estimator of A. What is its variance?

(5) Find the BLUE estimator of A, and its variance.

(6) Compare the 3 estimators. Explain!

Solution

$$p(\boldsymbol{x}, A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp^{-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}(x[n]-A)^2} \tag{3}$$

$$\frac{\partial^2 lnp}{\partial A^2} = -\frac{1}{\sigma^2}\sum_{i=0}^{N-1} r^{2n} \tag{4}$$

$$-E[\frac{\partial^2 lnp}{\partial A^2}] = -\frac{1}{\sigma^2}\sum_{i=0}^{N-1} r^{2n} \tag{5}$$

Hence

$$Var(\hat{A}) \geq \frac{\sigma^2}{\sum_{i=0}^{N-1} r^{2n}} \tag{6}$$

To show that an efficient estimator exists,

$$\begin{aligned}
\frac{\partial lnp}{\partial A} &= \frac{1}{\sigma^2}(\sum_{i=0}^{N-1} x[n]r^n - A\sum_{i=0}^{N-1} r^{2n}) && (7)\\
&= (\frac{\sum_{i=0}^{N-1} r^{2n}}{\sigma^2})(\frac{\sum_{i=0}^{N-1} x[n]r^n}{\sum_{i=0}^{N-1} r^{2n}} - A) && (8)\\
&= I(A)(\hat{A} - A) && (9)
\end{aligned}$$

$\hat{A} = \frac{\sum_{i=0}^{N-1} x[n]r^n}{\sum_{i=0}^{N-1} r^{2n}}$ is efficient and $1/I(A)$ is the variance.

The linear model is of the form $\boldsymbol{x} = \boldsymbol{h}A + \boldsymbol{w}$, where

$$\boldsymbol{h} = [1, r, r^2, \cdots, r^{N-1}]^t \tag{10}$$

From the linear model, the MVU estimator of A is

$$\hat{A} = (\boldsymbol{h}^t\boldsymbol{h})^{-1}\boldsymbol{h}^t\boldsymbol{x} \tag{11}$$

and its variance is given by $\sigma^2_{\hat{A}} = \sigma^2(\boldsymbol{h}^t\boldsymbol{h})^{-1}$ We can compute

$$\boldsymbol{h}^t\boldsymbol{h} = \sum_{n=0}^{N-1} r^{2n} \tag{12}$$

$$\boldsymbol{h}^t\boldsymbol{x} = \sum_{n=0}^{N-1} r^n x[n] \tag{13}$$

Hence,

$$\hat{A}_{MVU} = \frac{\sum_{i=0}^{N-1} x[n]r^n}{\sum_{i=0}^{N-1} r^{2n}}. \tag{14}$$

and

$$\sigma^2_{\hat{A}} = \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}}. \tag{15}$$

The BLUE estimator of A is given by

$$\hat{A}_{BLUE} = \frac{\boldsymbol{s}^t C^{-1} \boldsymbol{x}}{\boldsymbol{s}^t C^{-1} \boldsymbol{s}}, \tag{16}$$

and its variance

$$\sigma_{\hat{A}_{BLUE}} = \frac{1}{\boldsymbol{s}^t C^{-1} \boldsymbol{s}}, \tag{17}$$

where here $C = \sigma^2 I$ (because the noise is white with variance σ^2), $\boldsymbol{s}$ is such that $E[\boldsymbol{x}] = \boldsymbol{s}A$; thus, $\boldsymbol{s} = [1 r \cdots r^{N-1}]^t$. Replacing in (16), we obtain

$$\begin{aligned} \hat{A}_{BLUE} &= \frac{1/\sigma^2 \sum_{n=0}^{N-1} r^n x[n]}{1/\sigma^2 \sum_{n=0}^{N-1} r^{2n}} & (18) \\ &= \frac{\sum_{n=0}^{N-1} r^n x[n]}{\sum_{n=0}^{N-1} r^{2n}} & (19) \end{aligned}$$

and the variance

$$\begin{aligned} \sigma_{\hat{A}_{BLUE}} &= \frac{1}{\boldsymbol{s}^t C^{-1} \boldsymbol{s}} & (20) \\ &= \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}} & (21) \end{aligned}$$

The estimators provided by the CRLB, linear model and BLUE are all equal in this case!

Problem 3

Consider the problem of line fitting: given the observations

$$x[n] = A + Bn + w[n], \quad n = 0, 1, \cdots, N-1.$$

where $w[n]$ is WGN. We want to estimate A and B. Hence, we have a vector parameter $\theta = \begin{pmatrix} A \\ B \end{pmatrix}$

(1) Is it a linear model? If yes, write it in vector form $(\mathbf{x} = H\theta + \mathbf{w})$.

(2) Find the MVU estimators of A and B (simplify the formula).

(3) Are the estimators found in (2) efficient? Find their variances.

Solution

[1] Yes the model is linear in both parameters A and B. Let $\boldsymbol{\theta} = [AB]^t$. The linear form is given by

$$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{w}, \tag{22}$$

where

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{pmatrix} \tag{23}$$

and $\boldsymbol{w} = [w[0]w[1]\cdots w[N-1]]^t$.

[2] The MVU estimator of $\boldsymbol{\theta}$ is then given by (and since the noise is white)

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{H}^t\boldsymbol{H})^{-1}\boldsymbol{H}^t\boldsymbol{x} \tag{24}$$

Simplifying, we obtain

$$\boldsymbol{H}^t\boldsymbol{H} = \begin{pmatrix} N & \sum_{n=0}^{N-1} n \\ \sum_{n=0}^{N-1} n & \sum_{n=0}^{N-1} n^2 \end{pmatrix} = \begin{pmatrix} N & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{N(N-1)(2N-1)}{6} \end{pmatrix} \tag{25}$$

and

$$(\boldsymbol{H}^t\boldsymbol{H})^{-1} = \begin{pmatrix} \frac{2(2N-1)}{N(N+1)} & -\frac{6}{N(N+1)} \\ -\frac{6}{N(N+1)} & \frac{12}{N(N^2-1)} \end{pmatrix} \tag{26}$$

$$\boldsymbol{H}^t\boldsymbol{x} = \begin{pmatrix} \sum_{n=0}^{N-1} x[n] \\ \sum_{n=0}^{N-1} nx[n] \end{pmatrix} \tag{27}$$

It follows that the MVU estimators of A and B are

$$\hat{A}_{MVU} = \frac{2(2N-1)}{N(N+1)} \sum_{n=0}^{N-1} x[n] - \frac{6}{N(N+1)} \sum_{n=0}^{N-1} nx[n]$$

$$\hat{B}_{MVU} = -\frac{6}{N(N+1)} \sum_{n=0}^{N-1} x[n] + \frac{12}{N(N^2-1)} \sum_{n=0}^{N-1} nx[n]$$

[3] Yes, from the linear model theorem, the estimators are efficient and the variance of the estimator is given by

$$\boldsymbol{C}_{\hat{\boldsymbol{\theta}}} = \sigma^2(\boldsymbol{H}^t\boldsymbol{H})^{-1} \tag{28}$$

$$= \sigma^2 \begin{pmatrix} \frac{2(2N-1)}{N(N+1)} & -\frac{6}{N(N+1)} \\ -\frac{6}{N(N+1)} & \frac{12}{N(N^2-1)} \end{pmatrix} \tag{29}$$

Problem 4

Consider the problem of a DC level in white noise:

$$x[n] = A + w[n], \quad n = 0, 1, \cdots, N-1.$$

where $w[n]$ is white noise with variance σ^2 (*and of unspecified pdf*), then the problem is to estimate A. Since $w[n]$ is not necessarily Gaussian, we cannot use the MVU estimator formula [1]. Find the BLUE estimator of A.

<u>Hint</u>: Because $E[x[n]] = A, s[n] = 1$, and therefore $\mathbf{s} = \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

Solution

The BLUE estimator of A is given by

$$\hat{A}_{BLUE} = \frac{\boldsymbol{s}^t C^{-1} \boldsymbol{x}}{\boldsymbol{s}^t C^{-1} \boldsymbol{s}}, \tag{30}$$

where here $\boldsymbol{s} = [11 \cdots 1]^t$ and $C = \sigma^2 I$ (because white noise with variance σ^2). then, we obtain

$$\begin{aligned} \hat{A}_{BLUE} &= \frac{1/\sigma^2 \sum_{n=0}^{N-1} x[n]}{1/\sigma^2 N} &(31)\\ &= \frac{\sum_{n=0}^{N-1} x[n]}{N}. &(32) \end{aligned}$$

[1]Recall that the MVU estimator applies to a linear model with WGN.